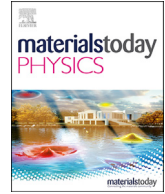




Contents lists available at ScienceDirect

Materials Today Physics

journal homepage: <https://www.journals.elsevier.com/materials-today-physics>

Graded thermal conductivity in 2D and 3D homogeneous hotspot systems



Chuang Zhang^a, Dengke Ma^b, Manyu Shang^c, Xiao Wan^a, Jing-Tao Lü^c, Zhaoli Guo^{a,*}, Baowen Li^{d,**}, Nuo Yang^{a,***}

^a State Key Laboratory of Coal Combustion, School of Energy and Power Engineering, Huazhong University of Science and Technology, Wuhan, 430074, China

^b NNU-SULI Thermal Energy Research Center (NSTER) and Center for Quantum Transport and Thermal Energy Science (CQTES), School of Physics and Technology, Nanjing Normal University, Nanjing, 210023, China

^c School of Physics and Wuhan National High Magnetic Field Center, Huazhong University of Science and Technology, Wuhan, 430074, PR China

^d Paul M Rady Department of Mechanical Engineering, Department of Physics, University of Colorado, Boulder, CO, 80309, USA

ARTICLE INFO

Article history:

Received 22 October 2021

Received in revised form

4 January 2022

Accepted 6 January 2022

Available online 14 January 2022

Keywords:

Hydrodynamic/ballistic phonon transport

Anomalous heat diffusion

Boltzmann transport equation

Non-Fourier heat conduction

ABSTRACT

Hotspot is a ubiquitous phenomenon in micro/nanoscale chips. Here, it is found that the thermal conductivity is not a constant in such a homogeneous system. The hotspots in homogeneous 2D disk/3D sphere and graphene disk are studied based on phonon Boltzmann transport equation. Instead a constant value, a graded thermal conductivity is observed even if the system size is fixed. The mechanisms of phonon scattering are analyzed. It is found that for a system with fixed size, the graded thermal conductivity is predictable as long as there is not sufficient phonon scattering, which is independent on material properties, dimensions or system size. This work may shed light on both theoretical and experimental studies on heat dissipation.

© 2022 Elsevier Ltd. All rights reserved.

1. Introduction

Thermal conductivity, which is a fundamental physical property of materials, is a constant that independent of system size and geometry in bulk materials. It is an intrinsic property that depends only on the component of materials. Heat conduction in such materials generally follows the Fourier's law, which implies that the heat carriers (phonons) undergo a diffusive process [1,2]. However, as the size or dimension of the system decreases, in particular when the size goes down to nanoscale and/or dimension is reduced to two dimension (2D) or quasi-one dimension (1D), there is still no rigorous mathematical proof that the Fourier's law is still valid. In contract, many researchers discovered that the thermal conductivity is a function of size and geometry [3–14].

The underlying physical mechanisms of non-Fourier heat

conduction mainly include: First, when the size of structures is comparable with the phonon mean free path, the phonon transport is largely affected by the boundary scattering [8,15], such that the thermal conductivity can be altered significantly by nano-engineering [7–11,16]. Second, a divergent thermal conductivity with system size was found in many low dimensional momentum conserved systems because of the existence of zero frequency and large wave length modes [17–20]. Third, as the system size is close or comparable to the phonon wave length, the wave nature of phonons is non-negligible in thermal transport [21,22]. Fourth, the possible existence of the second sound makes heat transfer like wave propagation [23–29]. This regime is usually called phonon hydrodynamic regime.

Most studies so far have focused on the length-dependent thermal conductivity [3,6,8–11,19,20]. In TDTR/Raman experiments of micro/nano scale heat conduction [30,31], researchers usually used an effective thermal conductivity to fit the experimental data. The quasi-ballistic thermal transport effects [32,33] are also measured with a nanoscale heat source comparable to the phonon mean free path [4,34–36]. The difference from Fourier's law is just the value of thermal conductivity depends on the size of

* Corresponding author.

** Corresponding author.

*** Corresponding author.

E-mail addresses: zlguo@hust.edu.cn (Z. Guo), Baowen.Li@Colorado.edu (B. Li), nuo@hust.edu.cn (N. Yang).

heat source. In these previous studies [30,31,35,36], the effective thermal conductivity is assumed as a constant for the whole structure and neglected the difference inside.

Recently, in a system with fixed size, an abnormal phenomenon - graded thermal conductivity - the thermal conductivity in the radial direction increases with the distance from the disk center, has been observed in homogeneous nanoscale graphene disk and carbon nanocone by molecular dynamics simulations [37,38]. Due to the limitation of computational resources, the diameter of system in previous molecular dynamics simulations [37,38] is below 25 nm. Does the graded thermal conductivity exist in a macro-system? What is the physical understanding on the mechanisms of graded thermal conductivity in homogeneous hotspot system?

In this work, the above mentioned questions are discussed by studying the graded thermal conductivity in homogeneous 2D disk/3D sphere (Figs. 1 and 2) with a fixed macroscopic size from ultra-low temperature to high temperature, without limiting to any specific material. The underlying physical mechanisms are to be analyzed by ballistic phonon transport, normal (N) scattering and resistive (R) scattering, respectively. The general conclusion will be exemplified by the graphene disk (Fig. 3).

The schematics of the 2D disk and 3D sphere are shown in Fig. 1(a) and Fig. 2(a), respectively, where the radii of the inner and outer heat baths are l and L , respectively. The temperatures of the inner and outer heat baths are fixed at $T_h = T_0 + \Delta T/2$ and $T_c = T_0 - \Delta T/2$, where $\Delta T/T_0 \rightarrow 0^+$. The local radial thermal conductivity κ is calculated by

$$\kappa(r) = -\frac{q(r)}{dT/dr}, \quad l < r < L, \quad (1)$$

where $q(r)$ is the local heat flux, namely the heat energy flow along

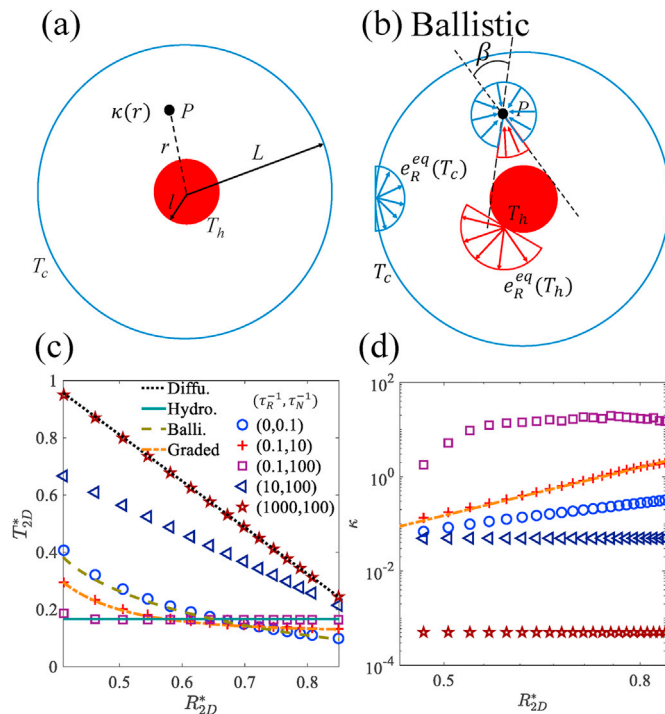


Fig. 1. (a) Schematic of homogeneous 2D disk. (b) Asymmetric phonon transport in the ballistic limit. For a fixed system size $L = 5l = 0.5$, the temperature profile (c) and graded thermal conductivity (d) along the radial direction with different scattering rates, where $T_{2D}^* = (T - T_c)/\Delta T$, $R_{2D}^* = (\ln(r/l) - 1)/(\ln(L/l) - 1)$, r is the distance from the disk center. More results and details are shown in SM 4–6 and TABLE S1.

the radial direction per unit area in a unit time. $T(r)$ is the local temperature, r is the distance from the center.

2. Model equation

The steady-state phonon Boltzmann transport equation (BTE) is used under the Callaway model and Matthiessen's rule [2,39–42], in which both the normal (N) scattering and resistive (R) scattering are included.

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} e = \tau_R^{-1} (e_R^{eq} - e) + \tau_N^{-1} (e_N^{eq} - e), \quad (2)$$

where e is the phonon distribution function of energy density, \mathbf{v} is the group velocity, \mathbf{x} is the spatial position. The heat flux and temperature in Eq. (1) are obtained by taking the moment of the distribution function. e_R^{eq} and e_N^{eq} are the associated phonon equilibrium distribution functions of energy density for R and N scattering, respectively. τ_R and τ_N are the relaxation times for R and N scattering, respectively. In the BTE simulations [2,40,42,43], the wave nature of phonons is not taken into account [21,22]. The distribution functions of all phonons emitting from the inner (or outer) heat bath are $e_R^{eq}(T_h)$ (or $e_R^{eq}(T_c)$) [40,42]. More details of phonon BTE and boundary conditions are shown in Supplemental Material (SM) 1.

The phonon transport will be simulated by solving phonon BTE numerically by the implicit discrete ordinate method [40,44]. In simulation of 2D disk/3D sphere with a fixed macroscopic size, the Debye approximation and gray model [2] are used, where no phonon dispersion and polarization are considered. Note that the heat conduction in 2D disk/3D sphere is not limited by specific materials properties so that all physical variables are dimensionless. The radii of inner and outer heat baths are fixed at $l = 0.1$ and $L = 0.5$, respectively (Figs. 1 and 2). The group velocity is $|\mathbf{v}| = 1$ and the specific heat is $C = 1$. The thermal effects of N (R) scattering on graded thermal conductivity will be investigated by adjusting the values of τ_N^{-1} or τ_R^{-1} .

In simulation of graphene disk (Fig. 3), the phonon dispersion and polarization of graphene are calculated using Vienna *ab initio* Simulation Package (VASP) combined with phonopy. And the effects of both frequency-dependent N and R scattering will be considered. More details on phonon properties of graphene and numerical solutions can be found in SM 2–3.

3. Results and discussion

The phonon transport in a homogeneous 2D disk with a fixed macroscopic system size is studied first. When the temperature increases from extremely low to high temperature, the phonon transport could go through the ballistic, hydrodynamic and diffusive regimes in turn [29,42,45]. In addition to numerical results, the analytical solutions in the ballistic [43] ($\tau_R^{-1} = 0$, $\tau_N^{-1} = 0$), diffusive ($\tau_R^{-1} \rightarrow \infty$, $\tau_N^{-1} = 0$) and hydrodynamic [23,24,26,27,46] ($\tau_N^{-1} \rightarrow \infty$, $\tau_R^{-1} = 0$) limits are also plotted in Fig. 1 to show the separate thermal effects of N or R scattering (Derivations of three limits are shown in SM 4).

In the ballistic regime [15,43] (e.g., $\tau_R^{-1} = 0$, $\tau_N^{-1} = 0.1$), phonon-phonon interaction/scattering can be totally neglected. As shown in Fig. 1(c)(d), the temperature profile is nonlinear and the radial thermal conductivity is not a constant anymore, instead it depends on r . The results are consistent with the analytical solutions in the ballistic limit [43,47] (see Fig. 1(b) or SM 4), i.e.,

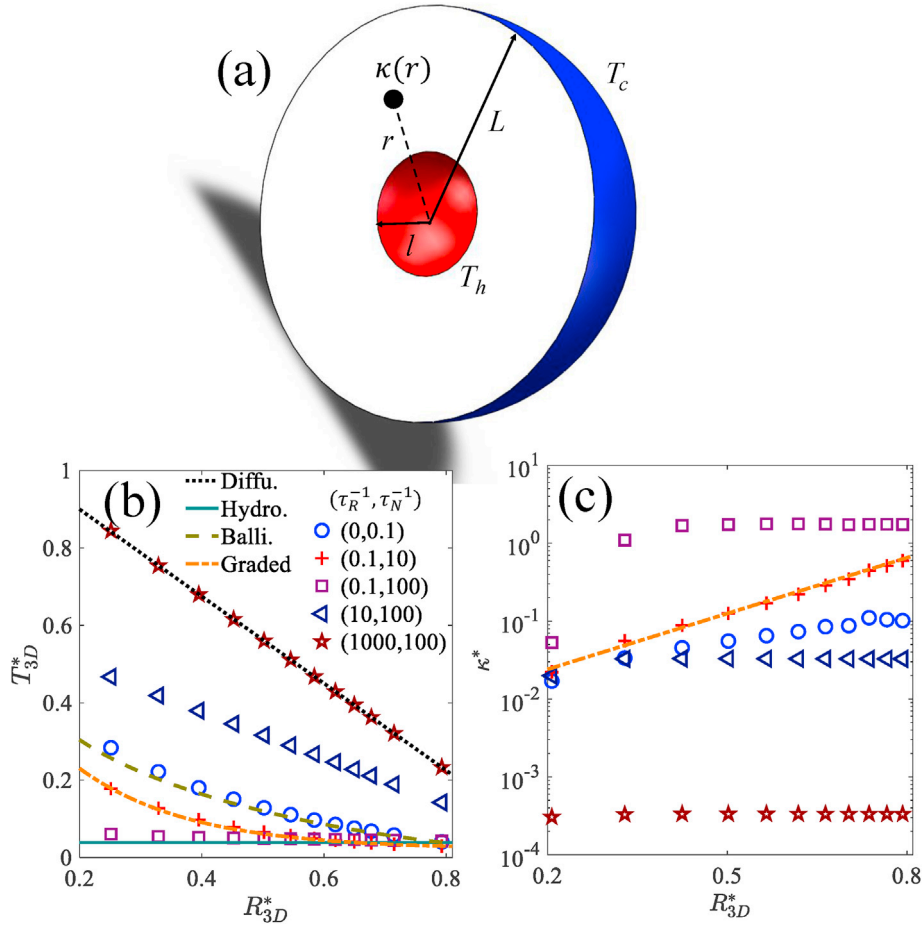


Fig. 2. (a) Schematic of 3D sphere. For a fixed system size ($L = 5l = 0.5$), the temperature profile (b) and graded thermal conductivity (c) along the radial direction with different scattering rates, where $T_{3D}^* = (T - T_c)/\Delta T$, $R_{3D}^* = (1/l - 1/r + 1)/(1/l - 1/L + 1)$, r is the distance from the sphere center. More results and details are shown in SM 4–6 and TABLE S2.

$$T(r) = \frac{2\arcsin(l/r)}{2\pi} T_h + \left(1 - \frac{2\arcsin(l/r)}{2\pi}\right) T_c. \quad (3)$$

This suggests the graded thermal conductivity, similar to what was observed in nanodisks [37] and nanocones [38] by molecular dynamics simulations.

In the phonon hydrodynamic regime, R scattering is very insufficient, and N scattering is much sufficient and dominates the heat conduction [23,24,26–28,46,48]. It can be observed that with the increase of τ_N^{-1} , the slopes of the numerical profiles of graded thermal conductivity in Fig. 1(d) increase first and then decrease gradually. As N scattering is much more sufficient than R scattering, the radial temperature goes to a constant and recovers the phonon hydrodynamic limit [23,24,26,27,46] (see SM 4), i.e.,

$$T = \frac{lT_h + LT_c}{l + L}. \quad (4)$$

At high temperature, R scattering starts to dominate the heat conduction so that phonon transport goes to the diffusive regime. It can be observed that with the increase of τ_R^{-1} , the temperature profile comes to linear and the graded thermal conductivity phenomenon disappears. The results agree well with the analytical solutions in the diffusive limit (see SM 4), i.e.,

$$dT \propto d \ln r. \quad (5)$$

Physical mechanisms.—As shown in Fig. 1, in homogeneous 2D disk with a fixed macroscopic size, the non-Fourier’s thermal transport phenomenon depends on scattering, i.e., τ_N^{-1} and τ_R^{-1} . In the following, the underlying physical mechanisms of phonon scattering are discussed in details.

Ballistic.—In the ballistic regime, phonon-phonon interaction/scattering rarely exists. Phonon migration dominates heat conduction [43,48]. For any point in the interior domain, phonons reach this point from the inner and outer thermal baths with different directions [13,43]. Both analytical (Eq. (3)) and numerical results predict that the temperature profile along radial direction has a non-linear dependence on the distance r in 2D disk (Fig. 1(c)). It is different from ballistic phonon transport in a symmetric system, in which the temperature is a constant [15]. In the symmetric system, all phonons emitting from one heat bath will be totally received by the other (see FIG. S5). So that the temperature gradient inside the system vanishes [15].

For ballistic transports in 2D disk, all phonons emitting from the inner bath will be received by the outer bath. However, phonons emitting from the outer heat bath will be received by both the inner and outer heat baths (see Fig. 1(b)). That means a portion of phonons are not received by the inner bath, which do not contribute to heat flux, but contribute to local energy or temperature. The temperature gradient is built by the asymmetric phonon migration, instead of phonon-phonon scattering. Because the heat flux from inner to outer is conserved, graded thermal conductivity can be

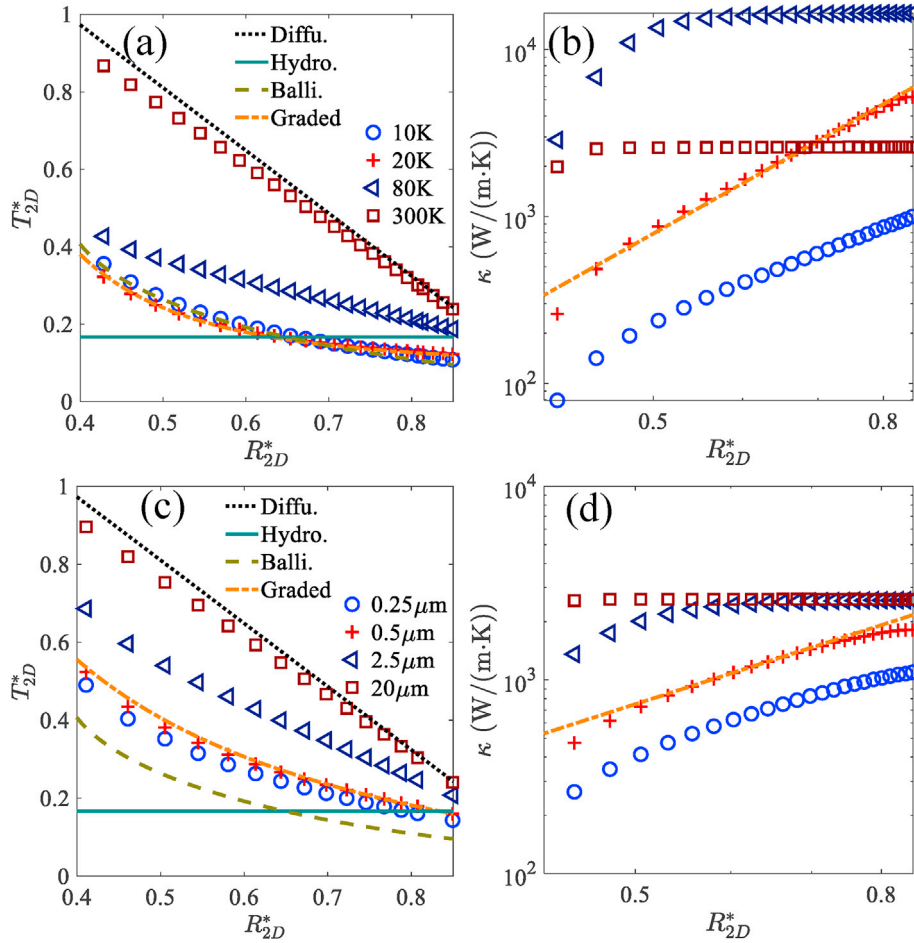


Fig. 3. The temperature profile and thermal conductivity in graphene disk [37]. (a) (b) Fixed size $L = 20 \mu\text{m}$. (c) (d) Fixed $T_0 = 300 \text{ K}$. More results and details are shown in SM 4–6 and TABLE. S3 and S4.

observed in 2D disk in the ballistic regime (Fig. 1(d)).

Scattering.—With the increase of temperature, phonon-phonon scattering dominates heat transfer [48]. In this case, the thermal effects of N scattering (τ_N^{-1} , momentum conserved) and R scattering (τ_R^{-1} , momentum not conserved) are discussed as follows.

N scattering.—In the phonon hydrodynamic regime, R scattering is very insufficient and N scattering dominates the phonon transport. N scattering does not cause thermal resistance [23,24,26–28,46], but affects energy distribution and temperature profile. When N scattering is insufficient ($\tau_N^{-1} \leq 10$), the graded thermal conductivity is attributed to inhomogeneous scattering (Fig. S9). This means that, because the hotspot blocks phonon's transport and affects phonon scattering, the N scattering is less near the hotspot, which limits the exchange of thermal energy. Instead, N scattering is more sufficient far from the center. As $\tau_N^{-1} \gg 10$, the N scattering is very sufficient inside the whole domain and goes to a constant temperature profile [23,24,46] (Eq. (4)), namely, graded thermal conductivity disappears. In a word, it can be observed that as N scattering increases, the graded thermal conductivity phenomenon increases first, and then fades away (Fig. 1(d)).

R scattering.—At high temperature, R scattering starts to play the leading role on heat conduction. Different from N scattering, R scattering does not conserve momentum, and causes thermal resistance [1,48]. With the increase of R scattering, the frequent energy exchange and heat dissipations decrease the temperature

jump near the heat baths [43] (Fig. S7). When R scattering is sufficient enough (e.g., $\tau_R^{-1} \geq 10$), there is a larger thermal resistance. And the heat conduction is close to or falls in the diffusive regime regardless of the strength of the N scattering. In the diffusive regime, Fourier's law is valid, so that there are a linear temperature profile and a constant thermal conductivity (Fig. 1(d)). In other words, in a structure with sufficient R scattering, no graded thermal conductivity appears.

Graded thermal conductivity.—Motivated by previous studies [37,38], an experimental formula of graded thermal conductivity is used to fit the numerical data approximately (Fig. 1(d)), i.e.,

$$\kappa(r) = \kappa_0 (R_{2D}^*)^\alpha, \quad R_{2D}^* = \frac{\ln(r/l) - 1}{\ln(L/l) - 1}, \quad (6)$$

where κ_0 is a constant, R_{2D}^* is the normalized coordination in 2D disk and α is the graded rate [37,38]. The detailed fitting parameters can be found in SM 5 and TABLE. S1. So that for a fixed disk size, there is no more homogenous value of thermal conductivity, instead a graded increasing thermal conductivity from the disk center to the outer.

For a given 2D disk with a fixed macroscopic size, the above results (Fig. 1) and analysis show that the graded thermal conductivity depends on the amount of phonon scattering. When phonon scattering is not sufficient and τ_N^{-1} and τ_R^{-1} are small, the temperature profiles are nonlinear and graded thermal

conductivity appears. In the ballistic regime, the graded thermal conductivity is caused by the asymmetric phonon migration [43] due to the spatial asymmetry of 2D disk as mentioned in Fig. 1(b) (or FIG. S5) and preceding paragraph (see *ballistic*). As the phonon-phonon scattering increases, the energy and momentum exchange among phonons break the asymmetric phonon migration gradually. However, the effects of N and R scattering on graded thermal conductivity are quite different (Fig. 1(d)). With the increase of τ_N^{-1} (see *N Scattering*), the graded phenomenon is enhanced first and then fades away due to diverging thermal conductivity [23,24,46]. As τ_R^{-1} increases (see *R Scattering*), the graded phenomenon fades away gradually.

3D sphere.—Does the graded thermal conductivity exist in 3D structures with a fixed macroscopic size? In order to look for the answer, the radial thermal conduction in a 3D sphere (Fig. 2(a)) is also investigated. The numerical results in different regimes are shown in Fig. 2(b)(c). It is found that the temperature profiles and thermal conductivities in 3D sphere are similar to those in the 2D disk. In addition, an exponential function of graded thermal conductivity is also used to fit the numerical data approximately (Fig. 2(c)), i.e.,

$$\kappa(r) = \kappa_0 \exp(\gamma R_{3D}^*), \quad R_{3D}^* = \frac{1/l - 1/r + 1}{1/l - 1/L + 1}, \quad (7)$$

where R_{3D}^* is the normalized coordination in 3D sphere and γ is a coefficient. The detailed fitting parameters can be found in SM 5 and TABLE. S2. Therefore, the graded thermal conductivity can appear in both 2D and 3D homogeneous hotspot systems with fixed macroscopic sizes.

Besides, the dimensional analysis [49] and more results of 2D disk/3D sphere are shown in SM 8.

Graphene disk.—Graphene, a very excellent thermal conductor widely studied [6,31,37], is picked up to illustrate.

Firstly, the size of graphene disk is fixed at $L = 20 \mu\text{m}$ and $l = 4 \mu\text{m}$. Then, the temperature T_0 is decreased gradually, as shown in Fig. 3(a)(b). As the temperature is changed from 300 K to 10 K, it can be observed that graded thermal conductivities and non-Fourier's phenomenon happen, which can be explained that the R scattering becomes insufficient, and the N scattering starts to dominate the heat transfer [26,27,40] as the temperature decreases. At $T_0 = 20$ K, the normalized temperature near the inner heat bath is even smaller than that in the ballistic limit, which is impossible if N scattering is insufficient. At ultra-low temperature (10 K), ballistic phonon transport dominates heat conduction so that the temperature profile recover the analytical solutions in the ballistic limit.

Secondly, the temperature of graphene disk is fixed at 300 K. Then, the system size L is decreased, as shown in Fig. 3(c)(d), where $L = 5l$. It can be observed that as system size decreases, the thermal conductivity along the radial direction is not a constant. Because as system size decreases, the ballistic phonon transport starts to play an important role on heat conduction [4,48,50–52]. It is noted that as the size of graphene disk is tens of nanometers, the graded thermal conductivity has been predicted by molecular dynamics [37,38], which is beyond the applications of phonon BTE [2]. According to the results of graphene disk (Fig. 3), it can be concluded that graded thermal conductivity occurs at low temperature or for a small sized system, which are consistent with the results in 2D disk/3D sphere.

4. Summary and conclusion

The graded thermal conductivity in homogeneous 2D disk/3D sphere and graphene disk with a spot heat source at the center is

studied by using the phonon Boltzmann transport equation. The results show that, for a homogenous hotspot system with fixed size, as long as phonon scattering is not sufficient, the thermal conductivity becomes inhomogeneous, namely, it increases from the center to the outer. This study may inspire a better understanding thermal transport of hotspots in electronics and chips.

Credit author statement

Chuang Zhang: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Funding acquisition. **Dengke Ma:** Methodology, Investigation, Formal analysis. **Manyu Shang:** Methodology, Investigation, Formal analysis. **Xiao Wan:** Investigation, Formal analysis. **Jingtao Lv:** Investigation, Formal analysis. **Zhaoli Guo:** Conceptualization, Supervision, Writing – original draft. **Baowen Li:** Conceptualization, Supervision, Writing – original draft. **Nuo Yang:** Conceptualization, Supervision, Writing – original draft, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work is supported by National Natural Science Foundation of China (12147122), National Key Research and Development Project of China No. 2018YFE0127800, the China Postdoctoral Science Foundation (2021M701565), Fundamental Research Funds for the Central Universities No. 2019kfyRCPY045 and Program for HUST Academic Frontier Youth Team. The authors thank the National Supercomputing Center in Tianjin (NSCC-TJ) and the China Scientific Computing Grid (SciGrid) for providing assistance in computations. The authors thanks the discussion with Prof. Qing Hao.

Appendix A. Supplementary data

Supplementary data to this article can be found online at <https://doi.org/10.1016/j.mtphys.2022.100605>.

References

- [1] M. Kaviani, *Heat Transfer Physics*, Cambridge University Press, 2008.
- [2] G. Chen, *Nanoscale Energy Transport and Conversion: A Parallel Treatment of Electrons, Molecules, Phonons, and Photons*, Oxford University Press, 2005.
- [3] Z. Zhang, Y. Ouyang, Y. Cheng, J. Chen, N. Li, G. Zhang, Size-dependent phononic thermal transport in low-dimensional nanomaterials, *Phys. Rep.* 860 (2020) 1–26.
- [4] P.G. Sverdrup, S. Sinha, M. Asheghi, S. Uma, K.E. Goodson, Measurement of ballistic phonon conduction near hotspots in silicon, *Appl. Phys. Lett.* 78 (21) (2001) 3331–3333.
- [5] N. Li, J. Ren, L. Wang, G. Zhang, P. Hänggi, B. Li, Colloquium: phononics: manipulating heat flow with electronic analogs and beyond, *Rev. Mod. Phys.* 84 (Jul 2012) 1045–1066.
- [6] X. Gu, Y. Wei, X. Yin, B. Li, R. Yang, Colloquium: phononic thermal properties of two-dimensional materials, *Rev. Mod. Phys.* 90 (Nov 2018) 41002.
- [7] D. Li, Y. Wu, P. Kim, L. Shi, P. Yang, A. Majumdar, Thermal conductivity of individual silicon nanowires, *Appl. Phys. Lett.* 83 (sep 2003) 2934–2936.
- [8] T.-K. Hsiao, H.-K. Chang, S.-C. Liou, M.-W. Chu, S.-C. Lee, C.-W. Chang, Observation of room-temperature ballistic thermal conduction persisting over 8.3 μm in SiGe nanowires, *Nat. Nanotechnol.* 8 (Jul 2013) 534–538.
- [9] V. Lee, C.-H. Wu, Z.-X. Lou, W.-L. Lee, C.-W. Chang, Divergent and ultrahigh thermal conductivity in millimeter-long nanotubes, *Phys. Rev. Lett.* 118 (Mar 2017) 135901.
- [10] C.W. Chang, D. Okawa, H. Garcia, A. Majumdar, A. Zettl, Breakdown of Fourier's law in nanotube thermal conductors, *Phys. Rev. Lett.* 101 (7) (2008) 75903.
- [11] X. Xu, L.F.C. Pereira, Y. Wang, J. Wu, K. Zhang, X. Zhao, S. Bae, C.T. Bui, R. Xie,

- J.T.L. Thong, B.H. Hong, K.P. Loh, D. Donadio, B. Li, B. Özyilmaz, Length-dependent thermal conductivity in suspended single-layer graphene, *Nat. Commun.* 5 (2014) 3689, 4.
- [12] G.D. Mahan, F. Claro, Nonlocal theory of thermal conductivity, *Phys. Rev. B* 38 (Jul 1988) 1963–1969.
- [13] G. Chen, Nonlocal and nonequilibrium heat conduction in the vicinity of nanoparticles, *J. Heat Tran.* 118 (3) (1996) 539–545.
- [14] K.T. Regner, A.J.H. McGaughey, J.A. Malen, Analytical interpretation of non-diffusive phonon transport in thermoreflectance thermal conductivity measurements, *Phys. Rev. B* 90 (Aug 2014) 64302.
- [15] A. Majumdar, Microscale heat conduction in dielectric thin films, *J. Heat Tran.* 115 (1) (1993) 7–16.
- [16] S.G. Volz, G. Chen, Molecular dynamics simulation of thermal conductivity of silicon nanowires, *Appl. Phys. Lett.* 75 (14) (1999) 2056–2058.
- [17] T.c. v. Prosen, D.K. Campbell, Momentum conservation implies anomalous energy transport in 1d classical lattices, *Phys. Rev. Lett.* 84 (Mar 2000) 2857–2860.
- [18] O. Narayan, S. Ramaswamy, Anomalous heat conduction in one-dimensional momentum-conserving systems, *Phys. Rev. Lett.* 89 (20) (2002) 200601.
- [19] S. Lepri, R. Livi, A. Politi, Thermal conduction in classical low-dimensional lattices, *Phys. Rep.* 377 (1) (2003) 1–80.
- [20] A. Dhar, Heat transport in low-dimensional systems, *Adv. Phys.* 57 (5) (2008) 457–537.
- [21] B. Yang, G. Chen, Partially coherent phonon heat conduction in superlattices, *Phys. Rev. B* 67 (May 2003) 195311.
- [22] D. Ma, A. Arora, S. Deng, G. Xie, J. Shiomi, N. Yang, Quantifying phonon particle and wave transport in silicon nanophononic metamaterial with cross junction, *Mater. Today Phys.* 8 (2019) 56–61.
- [23] R.A. Guyer, J.A. Krumhansl, Thermal conductivity, second sound, and phonon hydrodynamic phenomena in nonmetallic crystals, *Phys. Rev.* 148 (Aug 1966) 778–788.
- [24] R.A. Guyer, J.A. Krumhansl, Solution of the linearized phonon Boltzmann equation, *Phys. Rev.* 148 (Aug 1966) 766–778.
- [25] S. Lee, X. Li, Hydrodynamic phonon transport: past, present and prospects, in: *Nanoscale Energy Transport*, 2053–2563, IOP Publishing, 2020, 1–1.
- [26] S. Lee, D. Broido, K. Esfarjani, G. Chen, Hydrodynamic phonon transport in suspended graphene, *Nat. Commun.* 6 (1) (2015) 6290.
- [27] A. Cepellotti, G. Fugallo, L. Paulatto, M. Lazzeri, F. Mauri, N. Marzari, Phonon hydrodynamics in two-dimensional materials, *Nat. Commun.* 6 (1) (2015) 6400.
- [28] Y. Guo, M. Wang, Phonon hydrodynamics and its applications in nanoscale heat transport, *Phys. Rep.* 595 (2015) 1–44.
- [29] S. Huberman, R.A. Duncan, K. Chen, B. Song, V. Chiloyan, Z. Ding, A.A. Maznev, G. Chen, K.A. Nelson, Observation of second sound in graphite at temperatures above 100 K, *Science* 364 (6438) (2019) 375–379.
- [30] P. Jiang, X. Qian, R. Yang, Tutorial: time-domain thermoreflectance (TDTR) for thermal property characterization of bulk and thin film materials, *J. Appl. Phys.* 124 (oct 2018) 161103.
- [31] A.A. Balandin, S. Ghosh, W. Bao, I. Calizo, D. Teweldebrhan, F. Miao, C.N. Lau, Superior thermal conductivity of single-layer graphene, *Nano Lett.* 8 (3) (2008) 902–907.
- [32] C. Hua, A.J. Minnich, Heat dissipation in the quasiballistic regime studied using the Boltzmann equation in the spatial frequency domain, *Phys. Rev. B* 97 (Jan 2018) 14307.
- [33] X. Chen, C. Hua, H. Zhang, N.K. Ravichandran, A.J. Minnich, Quasiballistic thermal transport from nanoscale heaters and the role of the spatial frequency, *Phys. Rev. Applied* 10 (Nov 2018) 54068.
- [34] A.J. Minnich, J.A. Johnson, A.J. Schmidt, K. Esfarjani, M.S. Dresselhaus, K.A. Nelson, G. Chen, Thermal conductivity spectroscopy technique to measure phonon mean free paths, *Phys. Rev. Lett.* 107 (Aug 2011) 95901.
- [35] Y. Hu, L. Zeng, A.J. Minnich, M.S. Dresselhaus, G. Chen, Spectral mapping of thermal conductivity through nanoscale ballistic transport, *Nat. Nanotechnol.* 10 (Aug 2015) 701–706.
- [36] M.E. Siemens, Q. Li, R. Yang, K.A. Nelson, E.H. Anderson, M.M. Murnane, H.C. Kapteyn, Quasi-ballistic thermal transport from nanoscale interfaces observed using ultrafast coherent soft x-ray beams, *Nat. Mater.* 9 (Jan 2010) 26–30.
- [37] N. Yang, S. Hu, D. Ma, T. Lu, B. Li, Nanoscale graphene disk: a natural functionally graded material—how is Fourier's law violated along radius direction of 2D disk, *Sci. Rep.* 5 (2015) 14878.
- [38] D. Ma, H. Ding, X. Wang, N. Yang, X. Zhang, The unexpected thermal conductivity from graphene disk, carbon nanocone to carbon nanotube, *Int. J. Heat Mass Tran.* 108 (2017) 940–944.
- [39] J. Callaway, Model for lattice thermal conductivity at low temperatures, *Phys. Rev.* 113 (Feb 1959) 1046–1051.
- [40] Y. Guo, M. Wang, Heat transport in two-dimensional materials by directly solving the phonon Boltzmann equation under Callaway's dual relaxation model, *Phys. Rev. B* 96 (Oct 2017) 134312.
- [41] P.B. Allen, Analysis of nonlocal phonon thermal conductivity simulations showing the ballistic to diffusive crossover, *Phys. Rev. B* 97 (Apr 2018) 134307.
- [42] X.-P. Luo, Y.-Y. Guo, M.-R. Wang, H.-L. Yi, Direct simulation of second sound in graphene by solving the phonon Boltzmann equation via a multiscale scheme, *Phys. Rev. B* 100 (15) (2019) 155401.
- [43] H.-L. Li, B.-Y. Cao, Radial ballistic-diffusive heat conduction in nanoscale, *Nanos. Microsc. Therm.* 23 (1) (2019) 10–24.
- [44] C. Zhang, Z. Guo, S. Chen, An implicit kinetic scheme for multiscale heat transfer problem accounting for phonon dispersion and polarization, *Int. J. Heat Mass Tran.* 130 (2019) 1366–1376.
- [45] Z. Ding, J. Zhou, B. Song, V. Chiloyan, M. Li, T.-H. Liu, G. Chen, Phonon hydrodynamic heat conduction and Knudsen minimum in graphite, *Nano Lett.* 18 (1) (2018) 638–649. PMID: 29236507.
- [46] R. Yang, S. Yue, B. Liao, Hydrodynamic phonon transport perpendicular to diffuse-gray boundaries, *Nanos. Microsc. Therm.* 23 (1) (2019) 25–35.
- [47] D.B. Olfe, Application of a modified differential approximation to radiative transfer in a gray medium between concentric spheres and cylinders, *J. Quant. Spectrosc. Ra.* 8 (3) (1968) 899–907.
- [48] X. Li, S. Lee, Crossover of ballistic, hydrodynamic, and diffusive phonon transport in suspended graphene, *Phys. Rev. B* 99 (Feb 2019) 85202.
- [49] G.I. Barenblatt, *Dimensional Analysis*, CRC Press, 1987.
- [50] M.-H. Bae, Z. Li, Z. Aksamija, P.N. Martin, F. Xiong, Z.-Y. Ong, I. Knezevic, E. Pop, Ballistic to diffusive crossover of heat flow in graphene ribbons, *Nat. Commun.* 4 (2013) 1734.
- [51] G.F. Xie, Z.F. Ju, K.K. Zhou, X.L. Wei, Z.X. Guo, Y.Q. Cai, G. Zhang, Ultra-low thermal conductivity of two-dimensional phononic crystals in the incoherent regime, *npj Comput. Mater.* 4 (2018) 21.
- [52] X.L. Zhu, H.Y. Yang, W.X. Zhou, B.T. Wang, N. Xu, G.F. Xie, KAg_x (X = S, Se): High-Performance Layered Thermoelectric Materials for Medium-Temperature Applications, *ACS Appl. Mater. Interfaces* 12 (2020) 36102–36109.

Supplemental Material: Graded thermal conductivity in 2D and 3D homogeneous hotspot systems

Chuang Zhang^a, Dengke Ma^b, Manyu Shang^c, Xiao Wan^a, Jing-Tao Lü^c, Zhaoli Guo^{a,*}, Baowen Li^{d,*}, Nuo Yang^{a,*}

^aState Key Laboratory of Coal Combustion, School of Energy and Power Engineering, Huazhong University of Science and Technology, Wuhan, 430074, China

^bNNU-SULI Thermal Energy Research Center (NSTER) and Center for Quantum Transport and Thermal Energy Science (CQTES), School of Physics and Technology, Nanjing Normal University, Nanjing, 210023, China

^cSchool of Physics and Wuhan National High Magnetic Field Center, Huazhong University of Science and Technology, Wuhan 430074, P. R. China

^dPaul M Rady Department of Mechanical Engineering, Department of Physics, University of Colorado, Boulder, Colorado 80309, USA

1. Phonon Boltzmann transport equation

In this section, the stationary phonon Boltzmann transport equation (BTE) under the Callaway's dual relaxation model [1, 2, 3, 4] is introduced in detail.

This model equation is [1, 2, 5, 6, 3, 4, 7],

$$\mathbf{v} \cdot \nabla f = \frac{f_R^{eq} - f}{\tau_R} + \frac{f_N^{eq} - f}{\tau_N}, \quad (\text{S1})$$

where $f = f(\mathbf{x}, \mathbf{K}, \omega, p)$ is the phonon distribution function, \mathbf{x} is the physical position, \mathbf{K} is the wave vector and assumed to be isotropic, i.e., $\mathbf{K} = |\mathbf{K}|\mathbf{s}$, \mathbf{s} is the unit directional vector, ω is the angular frequency, $\mathbf{v} = \nabla_{\mathbf{K}}\omega$ is the group velocity, p is the phonon polarization. The left side of Eq. (S1) represents the phonon migration and the right side of Eq. (S1) is the phonon scattering term [8], which is composed of two parts: the first part is the momentum destroying resistive (R) scattering and the second part is the momentum conservation normal (N) scattering [5, 6, 3]. τ_R is the effective relaxation time of the R scattering, which is a combination of all momentum destroying phonon scattering except the boundary scattering based on the Mathiessen's rule [9]. f_R^{eq} is the associated equilibrium state of the R scattering and satisfies Bose-Einstein distribution [9], i.e.,

$$f_R^{eq}(T) = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}, \quad (\text{S2})$$

where k_B and \hbar are the Boltzmann constant and Planck constant reduced by 2π , T is the temperature. Different from R scattering, the N scattering satisfies the momentum conservation. Its displaced equilibrium

*Corresponding author

Email addresses: zlguo@hust.edu.cn (Zhaoli Guo), Baowen.Li@Colorado.edu (Baowen Li), nuo@hust.edu.cn (Nuo Yang)

distribution function and the effective relaxation time are f_N^{eq} and τ_N , respectively, where

$$f_N^{eq}(T, \mathbf{u}) = \frac{1}{\exp\left(\frac{\hbar\omega - \hbar\mathbf{K}\cdot\mathbf{u}}{k_B T}\right) - 1}, \quad (\text{S3})$$

where \mathbf{u} is the drift velocity.

Usually, the phonon BTE (Eq. (S1)) can be rewritten into a deviational energy form as below

$$\mathbf{v} \cdot \nabla e = \frac{e_R^{eq} - e}{\tau_R} + \frac{e_N^{eq} - e}{\tau_N}, \quad (\text{S4})$$

where the associated deviational distribution functions of energy density are [4, 3]

$$e = \frac{\hbar\omega D(f - f_R^{eq}(T_0))}{A}, \quad (\text{S5})$$

$$e_R^{eq} = \frac{\hbar\omega D(f_R^{eq} - f_R^{eq}(T_0))}{A}, \quad (\text{S6})$$

$$e_N^{eq} = \frac{\hbar\omega D(f_N^{eq} - f_R^{eq}(T_0))}{A}, \quad (\text{S7})$$

where T_0 is the reference temperature, $D(\omega, p)$ is the phonon density of state [9, 3]. In 2D systems, $A = 2\pi$, $D = |\mathbf{K}|/(2\pi|\mathbf{v}|)$, in 3D systems, $A = 4\pi$, $D = |\mathbf{K}|^2/(2\pi^2|\mathbf{v}|)$.

In this study, assuming a small temperature difference and a small drift velocity, i.e., $\Delta T/T_0 \ll 1$, $\mathbf{K} \cdot \mathbf{u} \ll \omega$, so that the equilibrium distribution function can be linearized, i.e.,

$$e_R^{eq}(T) \approx C(T - T_0)/A, \quad (\text{S8})$$

$$e_N^{eq}(T, \mathbf{u}) \approx C(T - T_0)/A + CT \frac{\mathbf{K} \cdot \mathbf{u}}{A\omega}, \quad (\text{S9})$$

where $C = C(\omega, p, T_0)$ is the mode specific heat at T_0 , i.e.,

$$C(\omega, p, T_0) = A \left. \frac{\partial e_R^{eq}}{\partial T} \right|_{T=T_0}. \quad (\text{S10})$$

The local deviational energy E , local temperature T and local heat flux \mathbf{q} can be updated by taking the moment of the distribution function, i.e.,

$$E = \sum_p \int \int e d\Omega d\omega, \quad (\text{S11})$$

$$T = T_0 + \frac{\sum_p \int \int e d\Omega d\omega}{\sum_p \int \int C d\omega}, \quad (\text{S12})$$

$$\mathbf{q} = \sum_p \int \int \mathbf{v} e d\Omega d\omega, \quad (\text{S13})$$

where $d\Omega$ and $d\omega$ are the integral over the whole 2D (or 3D) solid angle space and frequency space.

The N and R scattering satisfy the energy conservation, i.e.,

$$0 = \sum_p \int \int \frac{e_N^{eq}(T_N) - e}{\tau_N} d\Omega d\omega, \quad (\text{S14})$$

$$0 = \sum_p \int \int \frac{e_R^{eq}(T_R) - e}{\tau_R} d\Omega d\omega, \quad (\text{S15})$$

where T_R and T_N are local pseudotemperatures, which are introduced to ensure the conservation principle of the phonon scattering. In addition, the N scattering satisfies the momentum conservation, i.e.,

$$0 = \sum_p \int \int \frac{\mathbf{K}}{\omega} \frac{e_N^{eq}(T_N, \mathbf{u}) - e}{\tau_N} d\Omega d\omega. \quad (\text{S16})$$

Based on Eqs. (S14), (S15) and (S16), the macroscopic variables T_R , T_N and \mathbf{u} can be obtained,

$$T_R = T_0 + \left(\sum_p \int \int \frac{e d\Omega}{\tau_R} d\omega \right) \times \left(\sum_p \int \int \frac{C}{\tau_R} d\omega \right)^{-1}, \quad (\text{S17})$$

$$T_N = T_0 + \left(\sum_p \int \int \frac{e d\Omega}{\tau_N} d\omega \right) \times \left(\sum_p \int \int \frac{C}{\tau_N} d\omega \right)^{-1}, \quad (\text{S18})$$

$$\mathbf{u} = \frac{B}{T_N \sum_p \int \int \frac{|\mathbf{K}|^2}{\omega^2} \frac{C}{\tau_N} d\omega} \sum_p \int \int \frac{\mathbf{K}}{\omega} \frac{e}{\tau_N} d\Omega d\omega, \quad (\text{S19})$$

where $B = 2$ for two-dimensional materials and $B = 3$ for three-dimensional materials. Please note that the drift velocity is nonzero in the hydrodynamics regime [1, 10, 11, 2, 3, 4].

In addition, boundary conditions also play an indispensable role on thermal conduction. Here, the thermalizing boundary condition [12, 4, 3] is used to deal with the inner and outer heat baths with a fixed temperature T_w , i.e.,

$$e(\mathbf{x}_b, \mathbf{s}) = e_R^{eq}(T_w) \quad (\text{or } f(\mathbf{x}_b, \mathbf{s}) = f_R^{eq}(T_w)), \quad \mathbf{s} \cdot \mathbf{n}_b > 0, \quad (\text{S20})$$

where \mathbf{n}_b is the unit normal vector of the boundary \mathbf{x}_b pointing to the computational domain. Equation (S20) indicates that the distribution functions of all phonons emitting from the inner (outer) heat bath are e_R^{eq} with fixed heat bath temperature T_h (T_c) [3, 4].

Based on Eqs. (S12) and (S13), the local thermal conductivity along the radial direction in homogeneous 2D disk/3D sphere can be obtained, i.e.,

$$\kappa(r) = -\frac{q(r)}{dT/dr}, \quad l < r < L, \quad (\text{S21})$$

where $q(r)$ is the local heat flux, namely, the heat energy flow along the radial direction per unit area in a unit time. r is the distance from the center in homogeneous 2D disk/3D sphere. Then the graded thermal conductivity in 2D disk/3D sphere in the ballistic, diffusive and phonon hydrodynamic [2, 7, 13, 11, 6, 5] regimes can be investigated numerically based on phonon BTE.

2. Phonon dispersion and scattering in graphene

The phonon dispersion and polarization of graphene are calculated using Vienna *ab initio* Simulation Package (VASP) combined with phonopy. The supercell size for both second and third order force constant calculation is $5 \times 5 \times 1$. The numbers of mesh points for reciprocal space sampling are $151 \times 151 \times 1$. The phonon-phonon interaction and related properties can be calculated using phono3py. The phonon dispersion

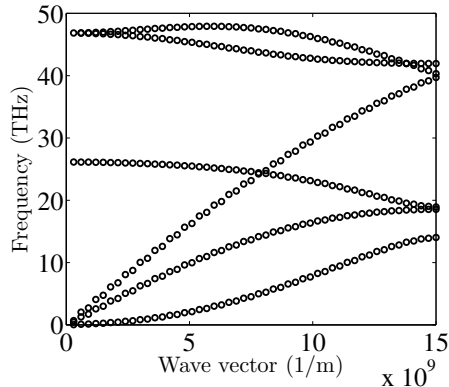


Figure S1: The phonon dispersion of graphene along Γ -M direction.

of graphene along Γ -M direction is shown in Fig. S1. The associated phonon N/R scattering rates along Γ -M direction in different temperatures are shown in Fig. S2, where the natural abundance of graphene is 1.1% ^{13}C .

In our present simulations, we need to discretize the whole spatial space and wave vector space to calculate the spatial distributions of the temperature and heat flux with finite system size. In order to reduce the computational amount, the isotropic wave vector space is assumed to save computational time. The phonon dispersion and thermal properties (Fig. S1, Fig. S2) of graphene along Γ -M direction are used, which is accurate enough to predict the thermal conductivity of graphene in a wide temperature range. The associated thermal conductivity of graphene in different temperatures are shown in Fig. S3, where the system size is infinite. The thermal conductivity predicted by Callaway's model is 2674 W/(m·K) and 17095 W/(m·K) in 300 K and 100 K, respectively.

3. Numerical solutions

The implicit discrete ordinate method (DOM) [3, 14] is used to solve the steady BTE, respectively. To ensure the numerical accuracy, enough cells are used to discretize the high-dimensional phase space.

In 2D disk, the spatial space is discretized with 200×200 uniform cells and the vanLeer limiter is used to ensure stability. For the solid angle space, we set $\mathbf{s} = (\cos \theta, \sin \theta)$, where $\theta \in [0, 2\pi]$ is the polar angle. Due to symmetry, the $\theta \in [0, \pi]$ is discretized with the N_θ -point Gauss-Legendre quadrature. The total number of the discretized directions is $2N_\theta$. In our 2D simulations, we set $N_\theta = 50$. In graphene disk, the discretized cells and directions are the same as those in 2D disk. The discretized phonon dispersion and polarization are shown in Fig. S1 and the mid-point rule is used for the numerical integration of the frequency space.

In 3D sphere, the spatial space is discretized with $200 \times 200 \times 200$ uniform cells and the vanLeer limiter is used, too. For the solid angle space, $\mathbf{s} = (\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi)$, where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ is the azimuthal angle. The $\cos \theta \in [-1, 1]$ is discretized with the N_θ -point Gauss-Legendre quadrature, while the azimuthal angular space $\varphi \in [0, \pi]$ (due to symmetry) is discretized with the $\frac{N_\varphi}{2}$ -point Gauss-Legendre

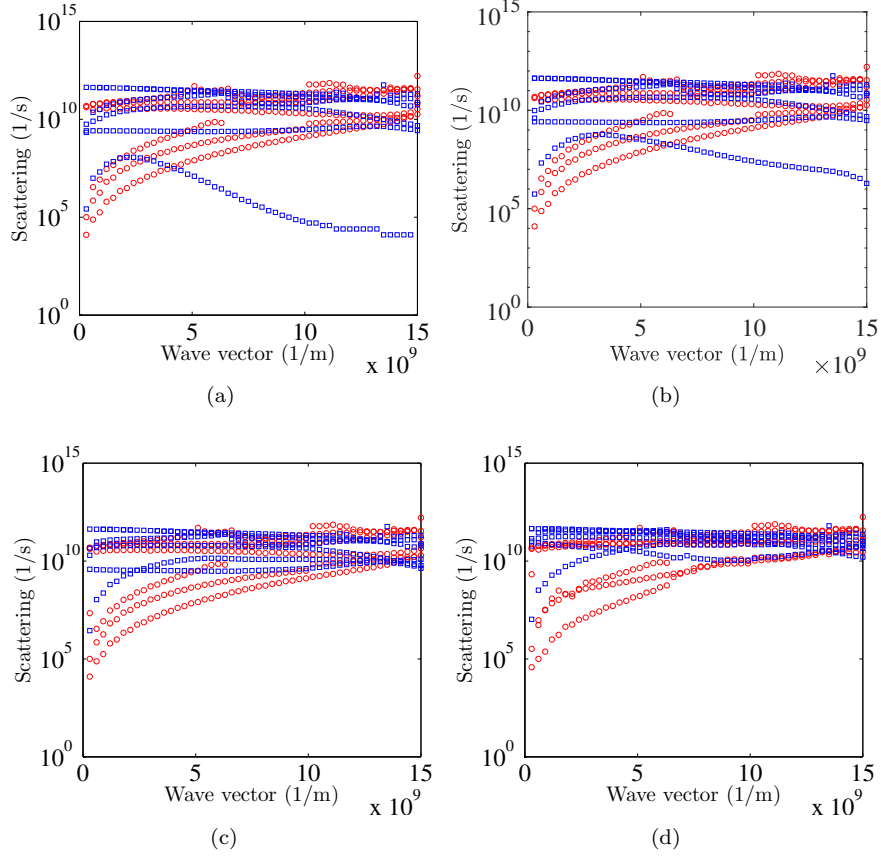


Figure S2: Phonon N/R scattering rates of graphene (natural abundance) along Γ -M direction in different temperatures, where the red circles represent R scattering rates and the blue squares represent N scattering rates. (a) 10 K, (b) 20 K, (c) 100 K, (d) 300 K.

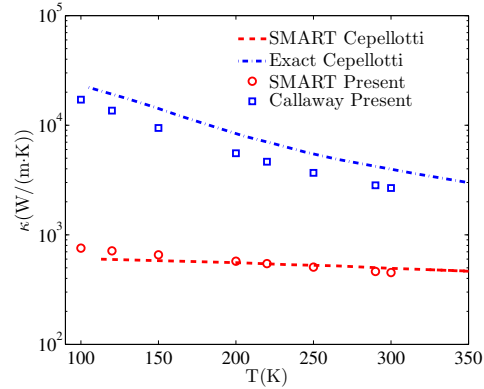


Figure S3: Thermal conductivity of graphene in different temperatures, where the system size is infinite. The symbols are the present results and the lines are the data obtained from Cepellotti's paper [5]. Exact represents the variational solution of phonon BTE under *ab initio* full scattering kernel [5], SMART represents the solution of Phonon BTE under single mode relaxation time approximation, Callaway represents the solution of Phonon BTE under Callaway's model [3, 7].

quadrature. In our 3D simulations, we set $N_\theta \times N_\varphi = 24 \times 24$ or 36×36 .

4. Analytical solutions of the phonon BTE in the ballistic, diffusive and hydrodynamic limits

The separate thermal effects of normal (N) scattering and resistive (R) scattering on graded thermal conductivity are investigated. The analytical solutions in three limits at steady state are derived based on phonon BTE (Eq.(S4)) with rigorous mathematical derivations:

1. ballistic limit (No N scattering, no R scattering),
2. diffusive limit (No N scattering, very frequent R scattering),
3. phonon hydrodynamic limit [2, 7, 13, 11, 6, 5] (No R scattering, very frequent N scattering).

Without special statements, the Debye approximation and gray model [9, 11] are used, i.e., $|\mathbf{v}| = \omega/|\mathbf{K}|$. The Knudsen number is introduced and defined as the ratio between the phonon mean free path $\lambda = |\mathbf{v}|\tau$ to the diameter of outer heat bath ($2L$), i.e., $\text{Kn} = \lambda/2L$, where $\tau^{-1} = \tau_N^{-1} + \tau_R^{-1}$. First, we derive the analytical solutions in 2D polar systems. Then the mathematical derivations in 3D spherical systems are similar and straightforward.

4.1. 2D disk

As the system size is much smaller than the phonon mean free path, i.e., $\text{Kn} \rightarrow \infty$, the heat conduction is in the ballistic regime [12, 15] and there is rare phonon-phonon intrinsic scattering. Equation (S4) at steady state in the ballistic regime can be written as follows

$$\mathbf{v} \cdot \nabla e = 0, \tag{S22}$$

which indicates that the phonon distribution function inside the system is independent of the physical position x . As shown in Figs. S4 and S5, for an arbitrary point P inside the system ($|OP| = r$), it can be found that when all phonons with arbitrary directions transporting through P , only a portion of them come from the inner hot heat bath with $e_R^{eq}(T_h)$ (between the angle formed by line PP_1 and PP_2), while the other come from the outer cold heat bath with $e_R^{eq}(T_c)$. Based on Eqs. (S12) and (S11), taking an integral of the distribution function in position P over the whole solid angle spaces leads to the temperature at $P(r)$ [12, 15], i.e., (Fig. S5)

$$T(r) = \frac{\beta}{2\pi} T_h + \left(1 - \frac{\beta}{2\pi}\right) T_c, \tag{S23}$$

where $\beta = 2 \arcsin(l/r)$. Note that if $l/L \rightarrow 1$, $l/r \rightarrow 1$ and $\beta \rightarrow \pi$. So that Eq. (S23) goes to a constant temperature, which is consistent with the results in the ballistic limit in symmetric system [16]. A comparison of the ballistic phonon transport in symmetric and asymmetric system is shown in Fig. S5. It can be found that the non-zero temperature gradient in the asymmetric system is built by asymmetric phonon migration, rather than phonon-phonon scattering.

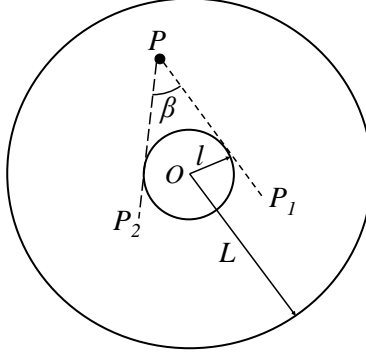


Figure S4: Heat conduction in homogeneous 2D disk in the ballistic limit at steady state. O is the center of the geometry. The radii of the inner and outer heat baths (or boundaries) are l , L , respectively. The temperatures of the inner and outer heat baths (boundaries) are set T_h and T_c . Lines PP_1 and PP_2 are tangent to the inner heat bath. $\beta = 2 \arcsin(l/r)$, where r is the distance from the disk center ($|OP|$).

Similarly, the heat flux along the radial direction is

$$\begin{aligned}
\mathbf{q}(r) \cdot \mathbf{n} &= \int_0^\beta |\mathbf{v}| \cos \zeta (e_R^{eq}(T_h) - e_R^{eq}(T_c)) d\zeta \\
&= \frac{2l}{r} |\mathbf{v}| (e_R^{eq}(T_h) - e_R^{eq}(T_c)) \\
&= \frac{l}{\pi r} |\mathbf{v}| C (T_h - T_c),
\end{aligned} \tag{S24}$$

where \mathbf{n} is the unit normal vector along the radial direction pointing from the inner to the outside. Then the thermal conductivity along the radial direction in the ballistic limit can be obtained by

$$\kappa(r) = -\frac{\mathbf{q}(r) \cdot \mathbf{n}}{dT/dr} = C |\mathbf{v}| \sqrt{(r^2 - l^2)}. \tag{S25}$$

As the system size is much larger than the phonon mean free path, i.e., $\text{Kn} \rightarrow 0$, here the phonon transport in the diffusive and phonon hydrodynamic limits [2, 7, 13, 11, 6, 5] are considered:

1. Heat conduction is in the diffusive limit (No N scattering, frequent R scattering).
2. Heat conduction is in the phonon hydrodynamic limit (No R scattering, frequent N scattering).

When there is no N scattering $\tau_N^{-1} = 0$ and frequent R scattering happens, the phonon BTE (Eq.(S4)) becomes

$$\mathbf{v} \cdot \nabla e = \frac{e_R^{eq} - e}{\tau_R}, \tag{S26}$$

and the heat conduction is in the diffusive limit. The distribution function can be approximated by the first-order Chapman-Enskog expansion as

$$e \approx e_R^{eq} - \tau_R \mathbf{v} \cdot \nabla e_R^{eq}. \tag{S27}$$

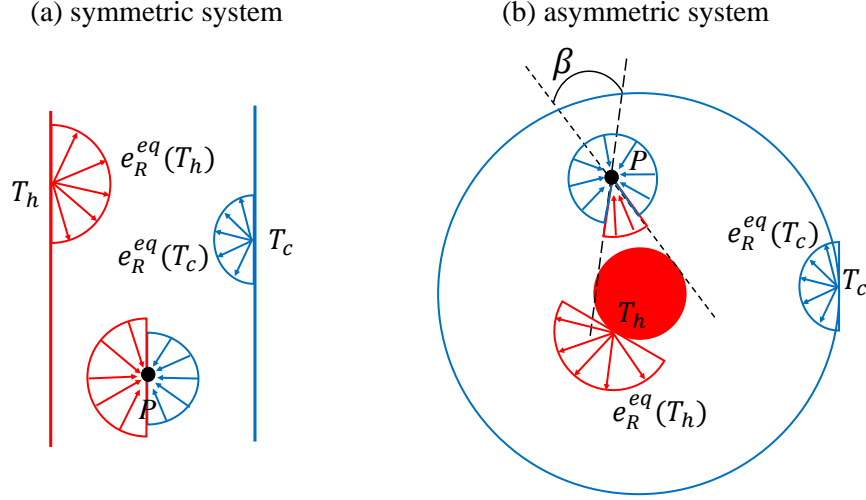


Figure S5: Ballistic phonon transport in (a) symmetric system [16] and (b) asymmetric system [17, 12] with two thermal baths (T_h , T_c). Based on isothermal boundary conditions (Eq. (S20)), the distribution functions of all phonons emitting from the hot and cold heat baths are $e_R^{eq}(T_h)$ and $e_R^{eq}(T_c)$, respectively [3, 4]. For an arbitrary position P in the interior domain, phonons going through this position come from the hot and cold thermal baths with different directions [17, 12]. (a) In the symmetric system, all phonons emitting from one heat bath will be totally received by the other. So that the temperature inside the system is a constant [16]. (b) In asymmetric system, for all phonons with arbitrary directions transporting through P , only a portion of them come from the hot heat bath with $e_R^{eq}(T_h)$, while the other come from the cold heat bath with $e_R^{eq}(T_c)$. So that the local temperature (Eq. (S23)) is not a constant, but related to $\beta = 2 \arcsin(l/r)$ (Fig. S4), where r is the distance between the disk center and position P .

Then we have

$$\begin{aligned}
\mathbf{q} &= \int_{2\pi} \mathbf{v} e d\Omega \\
&= - \int_{2\pi} \mathbf{v} \tau_R \mathbf{v} \cdot \nabla e_R^{eq} d\Omega \\
&= - \frac{\tau_R C |\mathbf{v}|^2}{2\pi} \int_{2\pi} \mathbf{s} \mathbf{s} \cdot \nabla T d\Omega \\
&= - \frac{1}{2} \tau_R C |\mathbf{v}|^2 \nabla T d\Omega, \\
&= -\kappa_{\text{diffusive}} \nabla T,
\end{aligned} \tag{S28}$$

where $\kappa_{\text{diffusive}} = \frac{1}{2} \tau_R C |\mathbf{v}|^2$. Due to the energy conservation, the total heat flux $Q(r)$ across the circle with radius r at steady state is a constant, where $Q = 2\pi r \mathbf{q} \cdot \mathbf{n}$. Then we have

$$\begin{aligned}
\text{constant} &= -2\pi r \kappa_{\text{diffusive}} dT/dr \\
\Rightarrow \text{constant} &= - \frac{1}{dT/d(\ln r)} \\
\Rightarrow dT &\propto d(\ln r).
\end{aligned} \tag{S29}$$

When there is no R scattering $\tau_R^{-1} = 0$ and frequent N scattering happens, the phonon BTE (Eq.(S4))

becomes

$$\mathbf{v} \cdot \nabla e = \frac{e_N^{eq} - e}{\tau_N}, \quad (\text{S30})$$

and the heat conduction is in the phonon hydrodynamic limit [7, 13, 11, 6, 5]. Due to energy and momentum conservation of N scattering [7, 13, 11], taking zero- and first- orders of moments of Eq. (S30) lead to

$$\int_{2\pi} \mathbf{v} \cdot \nabla e d\Omega = 0, \quad (\text{S31})$$

$$\int_{2\pi} \mathbf{v}\mathbf{v} \cdot \nabla e d\Omega = 0. \quad (\text{S32})$$

A zero-order approximation of the distribution function is made, i.e.,

$$e \approx e_N^{eq}. \quad (\text{S33})$$

So that Eq. (S32) becomes

$$\int_{2\pi} \mathbf{v}\mathbf{v} \cdot \nabla e_N^{eq} d\Omega = 0, \quad (\text{S34})$$

$$\int_{2\pi} |\mathbf{v}|^2 \mathbf{s}\mathbf{s} \cdot \nabla \left(\frac{C(T - T_0)}{2\pi} + \frac{CT}{2\pi} \frac{\mathbf{s} \cdot \mathbf{u}}{|\mathbf{v}|} \right) d\Omega = 0, \quad (\text{S35})$$

$$\int_{2\pi} |\mathbf{v}|^2 \mathbf{s}\mathbf{s} \cdot \nabla \left(\frac{C(T - T_0)}{2\pi} \right) d\Omega = 0, \quad (\text{S36})$$

$$\implies \nabla T = 0. \quad (\text{S37})$$

In other words, in the phonon hydrodynamic limit [7, 13, 11, 6, 5], the spatial divergence of temperature is zero. Namely, in homogeneous 2D disk, the temperature along the radial direction is a constant [11, 6, 5].

Considering the symmetry, the direction of the drift velocity \mathbf{u} is along the radial direction, i.e., $|\mathbf{u}| = \mathbf{u} \cdot \mathbf{n}$. When $l < r < L$, the total heat flux $Q(r)$ across the circle with radius r is

$$\begin{aligned} Q(r) &= 2\pi r \mathbf{n} \cdot \mathbf{q} \\ &= 2\pi r \mathbf{n} \cdot \left(\int_{2\pi} \mathbf{v} e_N^{eq} d\Omega \right) \\ &= 2\pi r \mathbf{n} \cdot \left(\int_{2\pi} \mathbf{v} \frac{CT}{2\pi} \frac{|\mathbf{K}| \mathbf{s} \cdot \mathbf{u}}{\omega} \right) \\ &= CT(r) r \mathbf{n} \cdot \int_{2\pi} \mathbf{s}\mathbf{s} \cdot \mathbf{u} d\Omega, \\ &= \pi C r T(r) |\mathbf{u}(r)|. \end{aligned} \quad (\text{S38})$$

Combining the boundary conditions, i.e., Eq. (S20), when $r \rightarrow l$,

$$\begin{aligned} Q(r) &= 2\pi l \mathbf{n} \cdot \left(\int_{\mathbf{s} \cdot \mathbf{n} > 0} \mathbf{v} e_R^{eq}(T_h) d\Omega + \int_{\mathbf{s} \cdot \mathbf{n} < 0} \mathbf{v} e_N^{eq} d\Omega \right) \\ &= 2lC|\mathbf{v}|T_h - 2lCT(r)|\mathbf{v}| + \frac{\pi}{2}CT(r)l|\mathbf{u}(r)|. \end{aligned} \quad (\text{S39})$$

When $r \rightarrow L$,

$$\begin{aligned} Q(r) &= 2\pi r \mathbf{n} \cdot \left(\int_{\mathbf{s} \cdot \mathbf{n} < 0} \mathbf{v} e_R^{eq}(T_c) d\Omega + \int_{\mathbf{s} \cdot \mathbf{n} > 0} \mathbf{v} e_N^{eq} d\Omega \right) \\ &= -2LC|\mathbf{v}|T_c + 2LC|\mathbf{v}|T(r) + \frac{\pi}{2}CT(r)L|\mathbf{u}(r)|. \end{aligned} \quad (\text{S40})$$

In addition, based on the energy conservation, $Q(r)$ is a constant at steady state. Combining above four equations, we have

$$T = \frac{lT_h + LT_c}{l + L}, \quad (\text{S41})$$

$$|\mathbf{u}(r)| = \frac{4|\mathbf{v}|lL(T_h - T_c)}{r\pi(lT_h + LT_c)}. \quad (\text{S42})$$

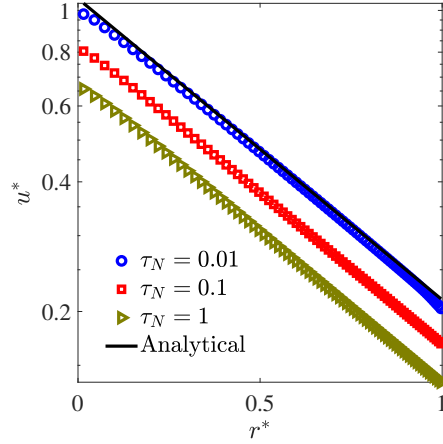


Figure S6: Numerical validation of Eq. (S42) in the phonon hydrodynamic regime in 2D disk model, where $\tau_R^{-1} = 0$, $r^* = \lg(r/l)/\lg(L/l)$, $u^* = (uT_0)/(|\mathbf{v}|\Delta T)$, u is the phonon drift velocity along the radial direction. Symbols: numerical solutions with different N scattering rates. Black line: analytical solutions (S42).

4.2. 3D sphere

For thermal conduction in 3D concentric sphere, the mathematical derivations are similar and straightforward. Similarly, the analytical solutions in 3D spherical systems can be derived:

In the ballistic limit,

$$T = \frac{T_h}{2} \left(1 - \sqrt{1 - l^2/r^2} \right) + \frac{T_c}{2} \left(1 + \sqrt{1 - l^2/r^2} \right), \quad (\text{S43})$$

$$\kappa(r) = \frac{C|\mathbf{v}|}{2} \sqrt{r^2 - l^2}. \quad (\text{S44})$$

In the diffusive limit,

$$dT \propto d(r^{-1}), \quad (\text{S45})$$

$$\kappa(r) = \frac{1}{3}C|\mathbf{v}|^2\tau_R. \quad (\text{S46})$$

Table S1: Fitting parameters of Eqs. (S53) and (S57) in 2D disk with fixed system size $L = 5l = 0.5$ (FIG. 1 in the manuscript).

$(\tau_R^{-1}, \tau_N^{-1})$	κ_0	α	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
(0.1, 10)	4.8	5.0	0.3	0.13

Table S2: Fitting parameters of Eqs. (S63) and (S65) in 3D sphere with fixed system size $L = 5l = 0.5$ (FIG. 2 in the manuscript).

$(\tau_R^{-1}, \tau_N^{-1})$	κ_0	γ	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
(0.1, 10)	0.008	5.5	0.23	0.029

In the phonon hydrodynamic [7, 13, 11, 6, 5] limit,

$$T = \frac{l^2 T_h + L^2 T_c}{l^2 + L^2}. \quad (\text{S47})$$

4.3. Graphene disk

If considering the phonon dispersion of graphene disk [18, 6, 5], the analytical solutions are similar to those in 2D disk. In the ballistic limit,

$$T(r) = \frac{\beta}{2\pi} T_h + \left(1 - \frac{\beta}{2\pi}\right) T_c \quad (\text{S48})$$

$$\kappa(r) = \sqrt{(r^2 - l^2)} \sum_p \int (|\mathbf{v}| C) d\omega. \quad (\text{S49})$$

In the diffusive limit,

$$dT \propto d(\ln r), \quad (\text{S50})$$

$$\kappa(r) = \frac{1}{2} \sum_p \int C |\mathbf{v}|^2 \tau_R d\omega. \quad (\text{S51})$$

In the phonon hydrodynamic limit [7, 13, 11, 6, 5],

$$T = \frac{lT_h + LT_c}{l + L}. \quad (\text{S52})$$

5. Graded thermal conductivity

In this section, the experimental formulas of graded thermal conductivity in 2D disk/3D sphere are used to fit the numerical data approximately.

In 2D/graphene disk [19, 20], the experimental formula is,

$$\kappa(r) = \kappa_0 (R_{2D}^*)^\alpha, \quad R_{2D}^* = \frac{\ln(r/l) - 1}{\ln(L/l) - 1}, \quad (\text{S53})$$

Table S3: Fitting parameters of Eqs. (S53) and (S57) in graphene disk with fixed system size $L = 20 \mu\text{m}$, $l = 4 \mu\text{m}$ (FIG. 3(a)(b) in the manuscript).

T_0 (K)	κ_0 (W/(m·K))	α	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
20	11000	3.8	0.38	0.12

Table S4: Fitting parameters of Eqs. (S53) and (S65) in graphene disk with fixed temperature $T_0 = 300$ K, where $L = 5l$ (FIG. 3(c)(d) in the manuscript).

L (μm)	κ_0 ($\text{W}/(\text{m}\cdot\text{K})$)	α	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
0.5	3000	2.0	0.53	0.16

where κ_0 is a constant for a fixed case, R_{2D}^* is the normalized coordination and α is the graded rate [19, 20]. In the non-diffusive regime, there is temperature jump near the heat source or heat sink. To remove this effect, here we ignore the spatial domain near the heat source or sink, for example, $R_{2D}^* \in [0.4, 0.85]$.

Based on the energy conservation, the associated temperature distributions along the radial direction can also be derived based on (S53), i.e.,

$$\begin{aligned}
2\pi r \kappa \frac{dT}{dr} &= \text{constant}, \\
\Rightarrow 2\pi \kappa \frac{dT}{d \ln(r)} &= \text{constant}, \\
\Rightarrow \kappa_0 (R_{2D}^*)^\alpha \frac{dT}{d \ln(r)} &= \text{constant}, \\
\Rightarrow (R_{2D}^*)^\alpha \frac{dT}{d(R_{2D}^*)} &= \text{constant}.
\end{aligned} \tag{S54}$$

When $\alpha \neq 1$, Eq. (S54) becomes

$$\frac{dT}{d((R_{2D}^*)^{1-\alpha})} = \text{constant}. \tag{S55}$$

When $\alpha = 1$, Eq. (S54) becomes

$$\frac{dT}{d(\ln(R_{2D}^*))} = \text{constant}. \tag{S56}$$

When $R_{2D}^* = 0.4$, the temperature is set as $T = T_{in}$, and when $R_{2D}^* = 0.85$, the temperature is set as $T = T_{out}$, where T_{in} and T_{out} are two coefficients obtained from phonon BTE. Then we can get the solutions of temperature along the radial direction based on Eqs. (S55) and (S56), i.e.,

$$T(R_{2D}^*) = c_1 (R_{2D}^*)^{1-\alpha} + c_2, \quad \alpha \neq 1, \tag{S57}$$

$$T(R_{2D}^*) = c_3 \ln(R_{2D}^*) + c_4, \quad \alpha = 1, \tag{S58}$$

where

$$c_1 = \frac{T_{out} - T_{in}}{0.85^{1-\alpha} - 0.4^{1-\alpha}}, \tag{S59}$$

$$c_2 = T_{in} - \frac{T_{out} - T_{in}}{2.125^{1-\alpha} - 1}, \tag{S60}$$

$$c_3 = \frac{T_{out} - T_{in}}{\ln 2.125}, \tag{S61}$$

$$c_4 = T_{in} - \frac{T_{out} - T_{in}}{\ln 2.125} * \ln(0.4). \tag{S62}$$

In 3D sphere, the experimental formula is,

$$\kappa(r) = \kappa_0 \exp(\gamma R_{3D}^*), \quad R_{3D}^* = \frac{1/l - 1/r + 1}{1/l - 1/L + 1}, \quad (\text{S63})$$

where R_{3D}^* is the normalized coordination in 3D sphere and γ is a constant for a fixed case. Here we ignore the spatial domain near the heat source or sink, for example, $R_{3D}^* \in [0.2, 0.8]$.

Based on the energy conservation, the associated temperature distributions along the radial direction can also be derived based on (S63), i.e.,

$$\begin{aligned} 4\pi r^2 \kappa \frac{dT}{dr} &= \text{constant}, \\ \Rightarrow \kappa_0 \exp(\gamma R_{3D}^*) \frac{dT}{d(1/r)} &= \text{constant}, \\ \Rightarrow \exp(\gamma R_{3D}^*) \frac{dT}{d(R_{3D}^*)} &= \text{constant}, \\ \Rightarrow \frac{dT}{d(\exp(-\gamma R_{3D}^*))} &= \text{constant}. \end{aligned} \quad (\text{S64})$$

When $R_{3D}^* = 0.2$, the temperature is set as $T = T_{in}$, and when $R_{3D}^* = 0.8$, the temperature is set as $T = T_{out}$, where T_{in} and T_{out} are two coefficients obtained from phonon BTE. Then we can obtain the temperature along the radial direction based on Eq. (S64), i.e.,

$$T(R_{3D}^*) = c_5 \exp(-\gamma R_{3D}^*) + c_6, \quad (\text{S65})$$

where

$$c_5 = \frac{T_{out} - T_{in}}{\exp(-0.8\gamma) - \exp(-0.2\gamma)}, \quad (\text{S66})$$

$$c_6 = T_{in} - \frac{T_{out} - T_{in}}{\exp(-0.6\gamma) - 1}. \quad (\text{S67})$$

The detailed fitting parameters of the graded thermal conductivity (FIG. 1-3 in the manuscript) are shown in TABLE S1,S2,S3,S4, respectively.

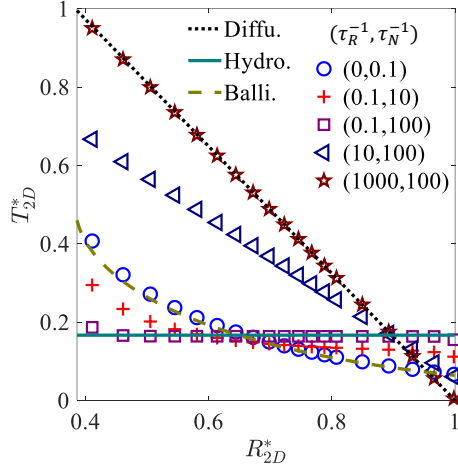
6. Spatial distributions of the temperature in the whole domain

In the non-diffusive regime, there is temperature jump near the heat source or heat sink. In order to remove the temperature jump effect, we ignore the spatial domain near the heat source or sink in the manuscript (Fig. 1-3). In this supplemental materials, the spatial distributions of the temperature in the whole domain are plotted to show the temperature jump at the heat sink/heat source, as shown in Fig. S7 and Fig. S8.

7. Inhomogeneous phonon scattering

When $(\tau_R^{-1}, \tau_N^{-1}) = (0.1, 10.0)$ in 2D disk model, the graded thermal conductivity is attributed to inhomogeneous scattering. This means that, because the hotspot blocks phonon's transport and affects phonon

(a) 2D disk model



(b) 3D sphere model

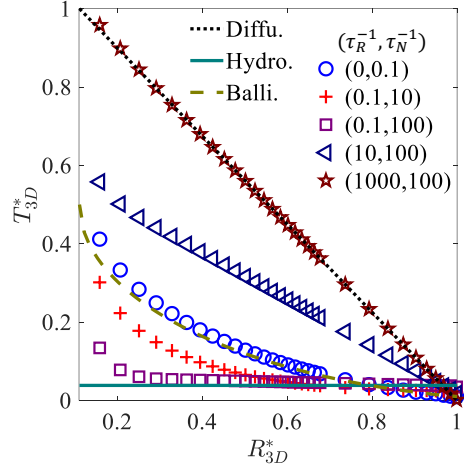
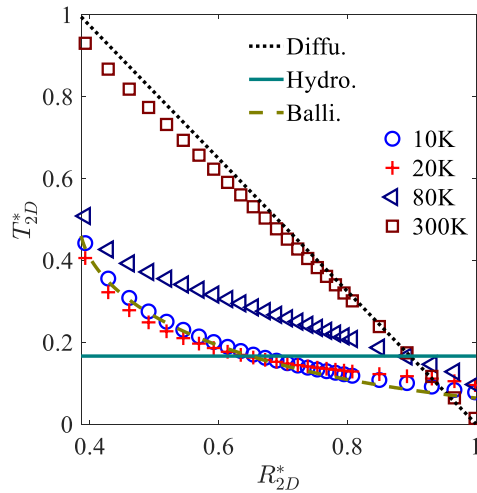


Figure S7: The spatial distributions of the temperature along the radial direction in 2D disk and 3D sphere model (Corresponding to Fig. 1c and Fig. 2b in the manuscript).

(a) Fix size, change temperature



(b) Fix temperature, change size

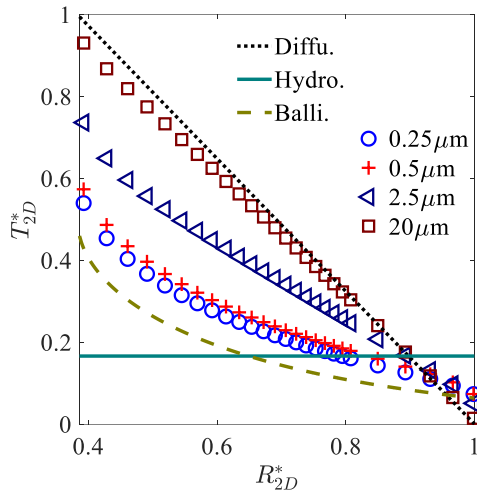


Figure S8: The spatial distributions of the temperature along the radial direction in graphene disk (Corresponding to Fig. 3a and Fig. 3c in the manuscript).

scatterings, the N scattering is less near the hotspot, which limits the exchange of thermal energy. Instead, N scattering is more sufficient far from the center, as shown in Fig. S9.

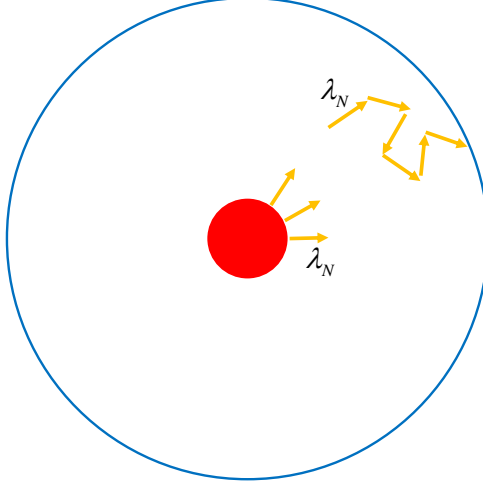


Figure S9: Schematic of inhomogeneous phonon scattering when $(\tau_R^{-1}, \tau_N^{-1}) = (0.1, 10.0)$ in 2D disk model, where $\lambda_N = |\mathbf{v}|\tau_N$ is the phonon mean free path of N scattering. More sufficient N scattering happens far from the hotspot center.

8. Dimensional analysis and numerical results

Table S5: Fitting parameters of Eqs. (S70) and (S57) in 2D disk with different (β_R, β_N) based on Eq. (S68) (Fig. S10).

(β_R, β_N)	κ_0	α	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
(0,10)	13.0	6.5	0.30	0.13

Table S6: Fitting parameters of Eqs. (S71) and (S65) in 3D sphere with different (β_R, β_N) based on Eq. (S68) (Fig. S11).

(β_R, β_N)	κ_0	γ	$(T_{in} - T_c)/(T_h - T_c)$	$(T_{out} - T_c)/(T_h - T_c)$
(0,10)	0.008	5.5	0.23	0.029

We choose C , T_0 , L , $|\mathbf{v}|$ as the reference variables to normalize the phonon BTE so that Eq. (S4) becomes

$$\frac{\mathbf{v}}{|\mathbf{v}|} \cdot \nabla_{\mathbf{x}^*} e^* = \beta_R (e_R^{eq,*} - e^*) + \beta_N (e_N^{eq,*} - e^*), \quad (\text{S68})$$

where the dimensionless parameters are

$$\begin{aligned} e^* &= \frac{e}{CT_0}, & x^* &= \frac{x}{2L}, & e_R^{eq,*} &= \frac{e_R^{eq}}{CT_0}, \\ e_N^{eq,*} &= \frac{e_N^{eq}}{CT_0}, & \beta_N &= \frac{2L}{|\mathbf{v}|\tau_N}, & \beta_R &= \frac{2L}{|\mathbf{v}|\tau_R}, \end{aligned} \quad (\text{S69})$$

where $2L$ is the diameter of the outer heat bath of 2D disk/3D sphere. Based on dimensional analysis [21], it can be observed that the heat conduction predicted by Eq. (S68) is totally decided by two dimensionless

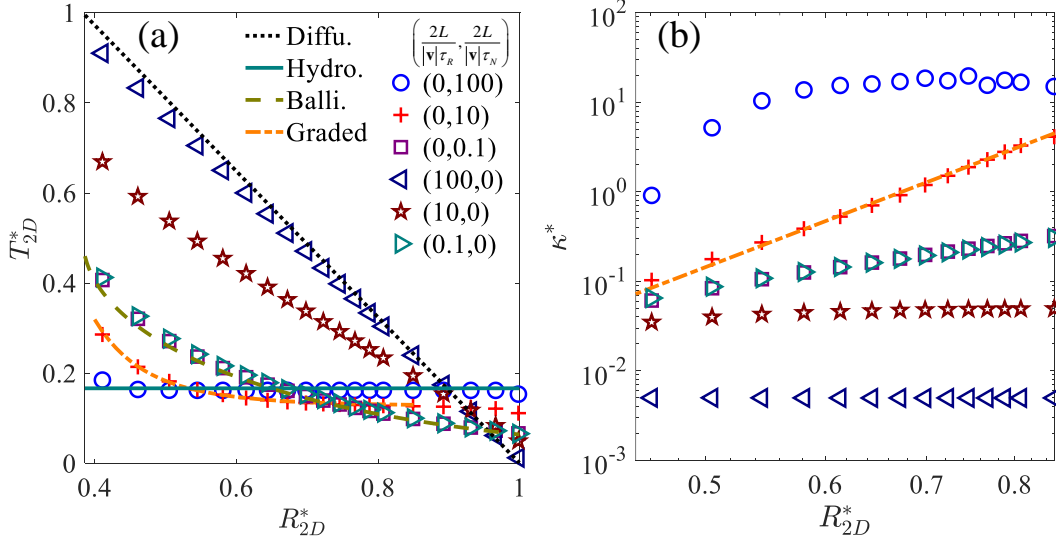


Figure S10: Thermal effects of N and R scattering are investigated individually in 2D disk with different $(\beta_R, \beta_N) = (2L/(|\mathbf{v}|\tau_R), 2L/(|\mathbf{v}|\tau_N))$ based on Eq. (S68). (a) Normalized temperature $T_{2D}^* = (T - T_c)/\Delta T$, $R_{2D}^* = (\ln(r/l) - 1)/(\ln(L/l) - 1)$. Only a portion of numerical results are plotted for better observations. Analytical solutions in the ballistic, diffusive and phonon hydrodynamic limits are shown in Eqs. (S23), (S29) and (S41), respectively. (b) Normalized thermal conductivity $\kappa^* = \kappa/(2CL|\mathbf{v}|)$. Symbols are numerical results and the orange dot line are the numerical fittings with power-law functions (Eq. S70). The detailed fitting parameters can be found in TABLE. S5.

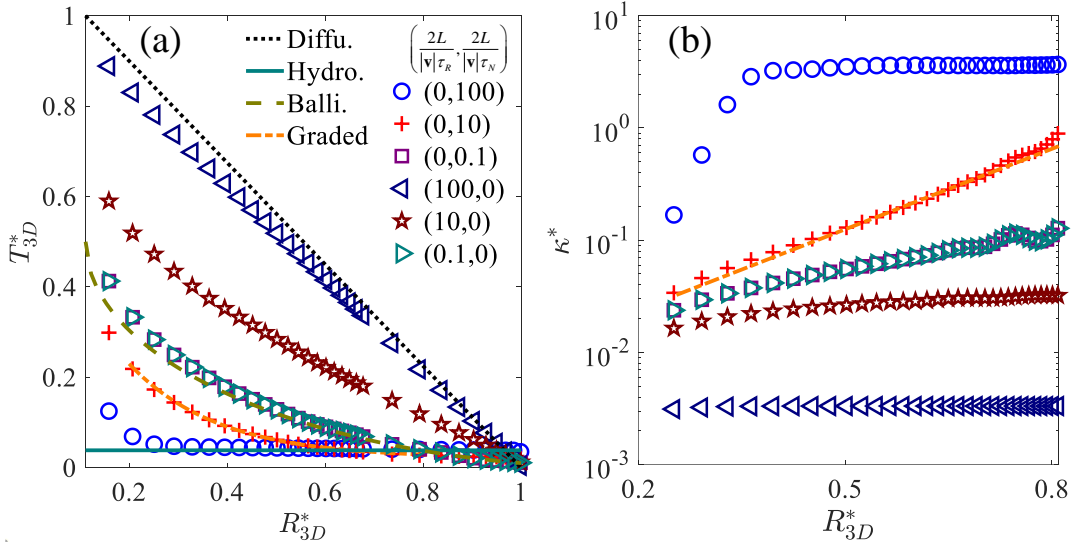


Figure S11: Thermal effects of N and R scattering are investigated individually in 3D sphere with different $(\beta_R, \beta_N) = (2L/(|\mathbf{v}|\tau_R), 2L/(|\mathbf{v}|\tau_N))$ based on Eq. (S68). (a) Normalized temperature $T_{3D}^* = (T - T_c)/\Delta T$, $R_{3D}^* = (1/l - 1/r + 1)/(1/l - 1/L + 1)$. Only a portion of numerical results are plotted for better observations. Analytical solutions in the ballistic, diffusive and phonon hydrodynamic limits are shown in Eqs. (S43), (S45) and (S47), respectively. (b) Normalized thermal conductivity $\kappa^* = \kappa/(2CL|\mathbf{v}|)$. Symbols are numerical results and the orange dot line are the numerical fittings with exponential functions (Eq. S71). The detailed fitting parameters can be found in TABLE. S6.

parameters, i.e., β_N and β_R , which represent the ratio between the diameter of the outer heat bath to the phonon mean free path of N and R scattering, respectively.

The thermal effects of N scattering (β_N) and R scattering (β_R) are investigated individually based on Eq. (S68). In simulation of 2D disk (Fig. S10) and 3D sphere (Fig. S11), the Debye approximation and gray model [9] are used, where no phonon dispersion and polarization are considered. Numerical results are consistent with the analytical solutions in the ballistic, diffusive and phonon hydrodynamic [7, 13, 11, 6, 5] limits. Furthermore, it can be observed that as β_N and β_R are small, the thermal conductivity along the radial direction is not a constant. Similarly, the numerical results (Figs. S10 and S11) of graded thermal conductivity in 2D disk/3D sphere are fitted with experimental formulas, i.e.,

$$\kappa^* = \kappa_0 (R_{2D}^*)^\alpha, \quad (\text{S70})$$

$$\kappa^* = \kappa_0 \exp(\gamma R_{3D}^*), \quad (\text{S71})$$

where the normalized thermal conductivity is $\kappa^* = \kappa/(2CL|v|)$. The detailed fitting parameters of the graded thermal conductivity in 2D disk/3D sphere are shown in TABLE S5 and S6, respectively.

References

- [1] M.-Y. Shang, C. Zhang, Z. Guo, and J.-T. Lü, “Heat vortex in hydrodynamic phonon transport of two-dimensional materials,” *Sci. Rep.*, vol. 10, p. 8272, may 2020.
- [2] X. Li and S. Lee, “Crossover of ballistic, hydrodynamic, and diffusive phonon transport in suspended graphene,” *Phys. Rev. B*, vol. 99, p. 085202, Feb 2019.
- [3] Y. Guo and M. Wang, “Heat transport in two-dimensional materials by directly solving the phonon Boltzmann equation under Callaway’s dual relaxation model,” *Phys. Rev. B*, vol. 96, p. 134312, Oct 2017.
- [4] X.-P. Luo, Y.-Y. Guo, M.-R. Wang, and H.-L. Yi, “Direct simulation of second sound in graphene by solving the phonon Boltzmann equation via a multiscale scheme,” *Phys. Rev. B*, vol. 100, no. 15, p. 155401, 2019.
- [5] A. Cepellotti, G. Fugallo, L. Paulatto, M. Lazzeri, F. Mauri, and N. Marzari, “Phonon hydrodynamics in two-dimensional materials,” *Nat. Commun.*, vol. 6, no. 1, p. 6400, 2015.
- [6] S. Lee, D. Broido, K. Esfarjani, and G. Chen, “Hydrodynamic phonon transport in suspended graphene,” *Nat. Commun.*, vol. 6, no. 1, p. 6290, 2015.
- [7] R. A. Guyer and J. A. Krumhansl, “Thermal conductivity, second sound, and phonon hydrodynamic phenomena in nonmetallic crystals,” *Phys. Rev.*, vol. 148, pp. 778–788, Aug 1966.

- [8] J. Y. Murthy, S. V. J. Narumanchi, J. A. Pascual-Gutierrez, T. Wang, C. Ni, and S. R. Mathur, “Review of multiscale simulation in submicron heat transfer,” *Int. J. Multiscale Computat. Eng.*, vol. 3, no. 1, pp. 5–32, 2005.
- [9] G. Chen, *Nanoscale energy transport and conversion: A parallel treatment of electrons, molecules, phonons, and photons*. Oxford University Press, 2005.
- [10] S. Lee and L. Lindsay, “Hydrodynamic phonon drift and second sound in a (20,20) single-wall carbon nanotube,” *Phys. Rev. B*, vol. 95, no. 18, p. 184304, 2017.
- [11] R. Yang, S. Yue, and B. Liao, “Hydrodynamic phonon transport perpendicular to diffuse-gray boundaries,” *Nanoscale Microsc. Therm.*, vol. 23, no. 1, pp. 25–35, 2019.
- [12] H.-L. Li and B.-Y. Cao, “Radial ballistic-diffusive heat conduction in nanoscale,” *Nanoscale Microsc. Therm.*, vol. 23, no. 1, pp. 10–24, 2019.
- [13] R. A. Guyer and J. A. Krumhansl, “Solution of the linearized phonon Boltzmann equation,” *Phys. Rev.*, vol. 148, pp. 766–778, Aug 1966.
- [14] C. Zhang, Z. Guo, and S. Chen, “An implicit kinetic scheme for multiscale heat transfer problem accounting for phonon dispersion and polarization,” *Int. J. Heat Mass Transfer*, vol. 130, pp. 1366–1376, 2019.
- [15] D. B. Olfe, “Application of a modified differential approximation to radiative transfer in a gray medium between concentric spheres and cylinders,” *J. Quant. Spectrosc. Ra.*, vol. 8, no. 3, pp. 899–907, 1968.
- [16] A. Majumdar, “Microscale heat conduction in dielectric thin films,” *J. Heat Transfer*, vol. 115, no. 1, pp. 7–16, 1993.
- [17] G. Chen, “Nonlocal and nonequilibrium heat conduction in the vicinity of nanoparticles,” *J. Heat Transfer*, vol. 118, no. 3, pp. 539–545, 1996.
- [18] E. Pop, V. Varshney, and A. K. Roy, “Thermal properties of graphene: Fundamentals and applications,” *MRS Bull.*, vol. 37, no. 12, pp. 1273–1281, 2012.
- [19] N. Yang, S. Hu, D. Ma, T. Lu, and B. Li, “Nanoscale graphene disk: a natural functionally graded material—how is Fourier’s law violated along radius direction of 2D disk,” *Sci. Rep.*, vol. 5, p. 14878, 2015.
- [20] D. Ma, H. Ding, X. Wang, N. Yang, and X. Zhang, “The unexpected thermal conductivity from graphene disk, carbon nanocone to carbon nanotube,” *Int. J. Heat Mass Transfer*, vol. 108, pp. 940 – 944, 2017.
- [21] G. I. Barenblatt, *Dimensional analysis*. CRC Press, 1987.